THE STRUCTURE OF CONTINUA

WITH INTERIOR POINTS

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PREFACE

The purpose of this paper is to examine the topological property of being a δ-subcontinuum of a compact continuum in a locally connected space S satisfying the weak separation axioms, as defined by O.H. Hamilton [5, pp.297-298], with special attention given to δ-indecomposable compact continua of S. (The numbers in square brackets,[ ], refer to the references in the Bibliography.)

The basic definitions and theorems are given in Chapter I. Of particular importance is the role played by the boundaries of the maximal connected open subsets (m.c.o.) of M, a compact continuum. It is shown that the boundary of a δ-subcontinuum of M completely determines the topological structure of the δ-subcontinuum. If S is completely separable there exists a δ-irreducible δ-subcontinuum of M about any connected set in M and this set is determined by the m.c.o. of M.

Chapter II treats mainly δ-indecomposable continua. A continuum M is decomposable if and only if it contains some proper subcontinuum with non-empty interior with respect to M. A δ-indecomposable continuum, on the other hand, may contain infinitely many proper subcontinua with non-empty interior. Even so, many properties of indecomposable continua carry over to δ-indecomposable continua. For example, under very mild conditions, if A is an indecomposable subcontinuum of the continuum M then $\delta(A)$, the δ-irreducible δ-subcontinuum of M containing A, is δ-indecomposable. However, examples are given to show that δ-indecomposability does not, in general, imply indecomposability.

In the third and last chapter the space S is restricted to Euclidean n-space. Every compact convex set with interior is seen to be δ-indecomposable. Also many results of Chapter II are strengthened by requiring the m.c.o. of M to be simply connected. Chapter III ends with a discussion of unsolved problems and conjectures.

I should like to acknowledge my indebtedness to Olan H. Hamilton for his valuable guidance in the preparation of this thesis; to the other members of my committee, Carl E. Marshall, Harrison S. Mendenhall, John E. Hoffman, Eugene K. McLachlan, and Paul Arthur; to L. Wayne Johnson for the teaching assitantship, often with reduced load, held the past three years; to the National Science Foundation for two summer fellowships; to Walter E. Stuermann for kindling my love for pure mathematics; and to my wife Betty for her encouragement at all times.

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CHAPTER I

GENERAL PROPERTIES OF δ-SUBCONTINUA

We shall assume throughout, unless explicitly stated otherwise, that the space S under consideration is locally connected and satisfies the weak separation axioms T_1 and T_2. Hence every singleton set in S is closed; and S is Hausdorff, as well as being locally connected at every point.

As for notation we shall use A+B for the set-theoretic union of A with B, AB for the set intersection of A and B, A-B for the set of elements belonging to A but not to B, Int(A) for the interior of A with respect to S, Cl(A) for the closure of A with respect to S, F(A) for the boundary of A relative to S, and 0 for the empty (or null) set.

In general when the words open, closed, or boundary occur we shall mean open, closed, or boundary with respect to S.

Our first result is used quite frequently in the forthcoming theorems in conjunction with local connectedness.

Lemma 1 [1, p.17]: Let A and B be subsets of S such that $AB \neq 0 \neq A-B$. If A is connected then A meets F(B).

Proof: If A fails to meet F(B) then A is the union of the two mutually separated sets AB and A-B. Consequently A would not be connected. This is a contradiction so A must meet F(B).

The first two theorems are structure theorems for subsets of S and depend heavily on local connectedness.

Theorem 1: If M is a subset of S then $\mathrm{Int}(M)=\Sigma_\alpha D_\alpha$ such that for each α, D_α is a maximal connected open (again relative to S) subset of M, and $D_\alpha D_\beta=0$ for $\alpha\neq\beta$.

Proof: Let x be an arbitrary element of Int(M), and D_x be the union of all the open connected subsets of M containing x. It is clear that the set D_x exists since S is locally connected at x. Then D_x is the maximal connected open subset of M containing x. If $\mathrm{Int}(M)=D_x$ there is nothing more to prove; otherwise let y belong to $\mathrm{Int}(M)-D_x$. As before, construct a maximal connected open subset D_y of M by forming the union of all the open connected subsets of M that contain y. Clearly D_x fails to meet D_y. By transfinite induction we obtain Int(M) equal $\Sigma_\alpha D_\alpha$ such that the D_α have the desired properties.

For brevity we shall often use the initials m.c.o. to stand for maximal connected open (with respect to S) subset.

Theorem 2: Let $\Sigma_\gamma D_\gamma$ be the union of any subcollection of the m.c.o. subsets of M, then $F(\Sigma_\gamma D_\gamma)=\mathrm{Cl}(\Sigma_\gamma F(D_\gamma))$.

Proof: We shall first show that $F(\Sigma_\gamma D_\gamma)$ is a subset of $\mathrm{Cl}(\Sigma_\gamma F(D_\gamma))$. Let x belong to $F(\Sigma_\gamma D_\gamma)$ and let V be any open set containing x. Since S is locally connected at x there exists an open set W containing x such that W is connected and contained in V. Then x in $F(\Sigma_\gamma D_\gamma)$ implies W meets both $\Sigma_\gamma D_\gamma$ and $S-\Sigma_\gamma D_\gamma$, so there is some D_W such that W meets both D_W and $S-D_W$. By Lemma 1, W must intersect $F(D_W)$ since W is connected, and consequently W meets $\Sigma_\gamma F(D_\gamma)$. But V contains W so V intersects $\Sigma_\gamma F(D_\gamma)$. Hence x belongs to $Cl(\Sigma_\gamma F(D_\gamma))$ so $Cl(\Sigma_\gamma F(D_\gamma))$ contains $F(\Sigma_\gamma D_\gamma)$.

We now show that $Cl(\Sigma_\gamma F(D_\gamma))$ is a subset of $F(\Sigma_\gamma D_\gamma)$. Let y belong to $Cl(\Sigma_\gamma F(D_\gamma))$ and let V be any open set containing y. We know V intersects $\Sigma_\gamma F(D_\gamma)$ so there is some D_V for which $F(D_V)$ meets V, and therefore $VD_V \neq 0$. However D_V is contained in $\Sigma_\gamma D_\gamma$ so V meets $\Sigma_\gamma D_\gamma$. Now suppose $V(S-\Sigma_\gamma D_\gamma)=0$. Then V is a subset of $\Sigma_\gamma D_\gamma$ so, for some γ, y belongs to D_γ. But y in $Cl(\Sigma_\gamma F(D_\gamma))$ and D_γ open about y imply D_γ meets $F(D_\beta)$ for some D_β, so $D_\gamma D_\beta \neq 0 \neq D_\gamma(S-D_\beta)$. Now the D_γ are disjoint so $D_\gamma D_\beta \neq 0$ implies $D_\gamma = D_\beta$. This, of course, is a contradiction since D_γ meets $S-D_\beta$. Consequently V intersects $S-\Sigma_\gamma D_\gamma$. We have shown that V intersects both $\Sigma_\gamma D_\gamma$ and $S-\Sigma_\gamma D_\gamma$, so y belongs to $F(\Sigma_\gamma D_\gamma)$ and therefore $F(\Sigma_\gamma D_\gamma)$ contains $Cl(\Sigma_\gamma F(D_\gamma))$.

Corollary: If $Int(M)=\Sigma_\alpha D_\alpha$, then $F(Int(M))=Cl(\Sigma_\alpha F(D_\alpha))$.

Definition 1: Let M be a compact continuum in S (that is M is connected and compact in the Bolzano-Weierstrass sense) and A be a subcontinuum of M. We shall say that A is a δ-subcontinuum of M if for each m.c.o. D of M, D is contained in A whenever D meets A or whenever F(D) is a subset of A.

One of the main features of a δ-subcontinuum is its invariance under homeomorphisms. We shall show this with the aid of the next two lemmas.

<u>Lemma 2:</u> Let M be a compact continuum in S and let f be a homeomorphism of S onto some space T. If D is an m.c.o. of f(M), then $F^{-1}(D)$ is an m.c.o. of M.

<u>Proof</u>: Since D is open and f is continuous we have $f^{-1}(D)$ open relative to S. Further D connected and f^{-1} continuous imply $f^{-1}(D)$ is connected. Hence $f^{-1}(D)$ is an open connected subset of M. If $f^{-1}(D)$ is not an m.c.o. of M then there exists an m.c.o. E of M such that $f^{-1}(D)$ is a proper subset of E. We then have D being a proper subset of f(E) with f(E) open and connected in f(M), since f continuous and E connected force f(E) to be connected and f^{-1} continuous with E open implies f(E) is open. But D is a maximal connected open subset of f(M). Consequently $f^{-1}(D)$ is an m.c.o. of M.

<u>Lemma 3</u>: Under the same hypothesis as for Lemma 2, $f^{-1}(F(D))$ equals $F(f^{-1}(D))$.

<u>Proof</u>: We first show that $f^{-1}(F(D))$ is a subset of $F(f^{-1}(D))$. Let x belong to $f^{-1}(F(D))$ and V be any open set containing x. Now f^{-1} continuous implies f(V) is open about f(x). Then f(x) is in F(D) so f(V) meets both D and T-D, and

V meets both $f^{-1}(D)$ and $f^{-1}(T-D)$. But $f^{-1}(T-D)=S-f^{-1}(D)$. Thus V meets $f^{-1}(D)$ and $S-f^{-1}(D)$, so x belongs to $F(f^{-1}(D))$. Consequently $F(f^{-1}(D))$ contains $f^{-1}(F(D))$.

To complete the proof we must have $f^{-1}(F(D))$ containing $F(f^{-1}(D))$. Let y belong to $F(f^{-1}(D))$ and W be any open set containing f(y). Now f is continuous so there exists an open set U containing y such that f(U) is contained in W. But y belonging to $F(f^{-1}(D))$ implies U meets both $f^{-1}(D)$ and $S-f^{-1}(D)$. Then W meets both D and T-D, so f(y) is in F(D). Therefore y is in $f^{-1}(F(D))$, and $F(f^{-1}(D))$ is a subset of $f^{-1}(F(D))$.

<u>Theorem 3</u>: Let f be a homeomorphism of S onto some space T. Further let M be a compact continuum in S, and A be a δ-subcontinuum of M. Then f(A) is a δ-subcontinuum of the compact continuum f(M).

<u>Proof</u>: It is well known that f(M) is a compact continuum and f(A) is a subcontinuum of f(M). Let D be an m.c.o. of f(M) such that D meets f(A). Now $f^{-1}(D)$ meets A with $f^{-1}(D)$ being an m.c.o. of M by Lemma 2; hence $f^{-1}(D)$ is a subset of A since A is a δ-subcontinuum of M. Then D must be contained in f(A). Now we consider the case of an m.c.o. D of f(M) such that F(D) is a subset of f(A). Then $f^{-1}(F(D))$ is a subset of A, so by Lemma 3, $F(f^{-1}(D))$ is contained in A. But, again by Lemma 2, $f^{-1}(D)$ is an m.c.o. of M and therefore $f^{-1}(D)$ is part of A by virtue of the fact that A is a δ-subcontinuum.

We have shown that for every m.c.o. D of f(M), D is contained in f(A) whenever D meets f(A) or F(D) is a subset of f(A). Consequently f(A) is a δ-subcontinuum of f(M).

Simple examples in the Euclidean plane show that it is not enough for f to be merely continuous and/or one-to-one, i.e. f(A) may not be a δ-subcontinuum of f(M) if A is a δ-subcontinuum of M and f is continuous and/or one-to-one. Of course f(A) is always a subcontinuum of the compact continuum f(M) if f is continuous.

We will be interested in ascertaining whether or not certain subsets of a compact continuum are δ-subcontinua of M. Our first result in this direction deals with pairs of subsets.

<u>Theorem 4</u>: Let A and B be closed subsets of the compact continuum M. If both A+B and AB are δ-subcontinua of M, then A and B are δ-subcontinua of M.

<u>Proof</u>: It is a known fact that A and B are subcontinua of M.

We shall now show that A is a δ-subcontinuum of M. Let D be an m.c.o. of M such that D meets A. Since A is part of A+B, D meets A+B and is then contained in A+B as A+B is a δ-subcontinuum of M. If D is not wholly contained in A, D must meet B-A. Then $DAB \neq 0$ since otherwise $D=D(A-B)+D(B-A)$ with A-B and B-A mutually separated, or D is not connected. Now AB is a δ-subcontinuum of M and $DAB \neq 0$ so D is contained in AB and therefore D is a subset of A. We have then shown that any m.c.o. D of M is a subset of A if D meets A.

Suppose D is any m.c.o. of M such that F(D) is contained in A. Now F(D) is contained in A+B, so D itself is contained in A+B since A+B is a δ-subcontinuum. If D meets AB then D is contained in AB (and hence in A) since AB is a δ-subcontinuum. If D fails to meet AB then either (1) $D=D(A-B)+D(B-A)$ with $D(A-B)\neq 0\neq D(B-A)$, or (2) D is contained in B-A, or (3) D is contained in A-B. However (1) implies D is not connected as we saw in the previous paragraph. Also (2) implies F(D) is contained in B so F(D) is contained in AB, and D is contained in AB since AB is a δ-subcontinuum. Then D misses B-A, so (2) cannot hold. Therefore (3) holds so D must be contained in A-B and hence in A. We have then shown that any m.c.o. D of M is a subset of A if F(D) is contained in A. Consequently A, and likewise B, is a δ-subcontinuum of M.

That the boundaries of maximal connected open subsets of a compact continuum M play an important role in the concept of δ-subcontinua of M can easily be seen from Definition 1. The next three theorems relate the boundaries of δ-subcontinua to each other and to the boundary of M.

Theorem 5: If A is a δ-subcontinuum of M then F(A) is contained in F(M).

Proof: Suppose x belongs to F(A)-F(M). Since x belongs to M-F(M) there exists an open set V containing x such that V is contained in M. Now S is locally connected so there is an open connected set W containing x such that W is a subset of V. Let D be the m.c.o. of M such that W is contained in D. We then have x in D and x in A (x is in F(A) and F(A) is part of A since A is closed). The subcontinuum A is a δ-subcontinuum of M so D is a subset of A. But x in F(A) and D open containing x imply D meets S-A. We have a contradiction so F(A)-F(M) is null. Consequently F(A) is contained in F(M).

Theorem 6: If A is a δ-subcontinuum of M and D is an m.c.o. of M such that F(D) is contained in A, then F(D) is contained in F(A).

Proof: Since A is a δ-subcontinuum of M we have D contained in A, or Cl(D) contained in A. Then, if F(D) meets S-F(A), F(D) can meet only A-F(A) (i.e. F(D) is a subset of F(A)+(A-F(A)). Suppose x belongs to F(D)(A-F(A)), so x belongs to Int(A). Then there exists an open set V containing x such that V is contained in A. Also S is locally connected so there is an open set W containing x such that W is connected and is a subset of V. Now x lies in F(D) so W meets D. Hence W+D is an open and connected subset of M. But x in F(D) also implies $W(S-D)\neq 0$, or (since W is contained in A) $W(A-D)\neq 0$. Consequently D is a proper subset of W+D. This is contradictory to the fact that D is a maximal connected open subset of M. Therefore (A-F(A)) F(D) is null so F(D) is contained in F(A).

Corollary 1: If A is a δ-subcontinuum of M and D is an m.c.o. of M such that D meets A, then F(D) is contained in F(A).

Proof: Since D meets A with D an m.c.o. of M and A a δ-subcontinuum, we have D as a subset of A. Now A is closed so Cl(D), and hence F(D), is contained in A. Hence F(D) is a subset of F(A) by the theorem.

Corollary 2: If D is an m.c.o. of M then F(D) is contained in F(M).

Proof: Clearly M is a δ-subcontinuum of itself and D meets M, so F(D) is contained in F(M) by Corollary 1.

Corollary 3: Let A be a δ-subcontinuum of M. Then F(A)=F(M) if and only if A=M.

Proof: Clearly A=M implies F(A)=F(M). Hence we need only show that F(A)=F(M) implies A=M. If A$\neq$M, then A must be a proper subset of M. Therefore let us suppose there is an element x belonging to M-A. Now M=Int(M)+F(M) with F(M) contained in F(A) (which is a subset of A) so x must belong to Int(M). By Theorem 1 let D be the m.c.o. of M such that x lies in D. By Corollary 2 above, F(D) is contained in F(M) with F(M)=F(A) so F(D) is contained in A. But A is a δ-subcontinuum of M so D is contained in A. Therefore x lies in A since x belongs to D. However we assumed that x belonged to M-A. Consequently M-A is empty, so A=M.

By changing Corollary 3 slightly we are led to believe that if A and B are distinct δ-subcontinua of M then A and B must differ in their boundaries, i.e., A and B must differ in more than just their interiors. The following theorem substantiates our belief.

Theorem 7: Let A and B be δ-subcontinua of M. For B to be a subset of A it is necessary and sufficient that F(B) be a subset of F(A).

Proof: If B is a subset of A then F(B) is a subset of F(A) (by an argument similar to the proof of Theorem 5). Conversely, if B is not contained in A there is some point x belonging to B-A. Since F(B) is contained in A, we have x belonging to Int(B). By Theorem 1, there exists an m.c.o. D of M such that x lies in D. Now B is a δ-subcontinuum of M so D is contained in B. Then, by Corollary 1 to Theorem 6, F(D) is contained in F(B). But F(B) is a subset of A and A is a δ-subcontinuum of M, so D is a subset of A and x must then belong to A. However x belongs to B-A by assumption. Consequently B-A is empty, so B is a subset of A.

Corollary: Let A and B be δ-subcontinua of M. If the symmetric difference of A and B is not empty then the symmetric difference of F(A) and F(B) is not empty.

Proof: If F(B)-F(A) were empty then B would be a subset of A by the theorem, so B-A would be empty. The same reasoning holds for F(A)-F(B).

Our next two theorems deal with well-ordered monotonic collections of δ-subcontinua of a compact continuum in a completely separable space.

Theorem 8: Let S be completely separable and let M be a compact continuum in S. If $\{A_\alpha\}$ is a well-ordered monotonic descending sequence of distinct δ-subcontinua of M then A, the intersection of the A_α, is a δ-subcontinuum of M.

Proof: It is well known that A is a non-vacuous subcontinuum of M. Let D be an m.c.o. of M such that D meets A. Then D meets each of the A_α, so D is contained in each A_α since all the A_α are δ-subcontinua of M. Hence D is contained in A. We have then shown that for each m.c.o. D of M such that D meets A, D is contained in A. Now suppose D is an m.c.o. of M such that F(D) is contained in A. Then F(D) is contained in each of the A_α, so D is contained in each A_α since all the A_α are δ-subcontinua of M. Therefore D is contained in A, so each m.c.o. D of M such that F(D) is contained in A is a subset of A. Consequently A is a δ-subcontinuum of M.

Theorem 9: Let $\{A_\alpha\}$ be a well-ordered monotonic ascending sequence of distinct δ-subcontinua of M and let A be the closure of the union of the A_α. Further suppose that M is such that if the boundary of an m.c.o. of M is contained in A then the boundary is contained in a finite number of A_α. Then A is a δ-subcontinuum of M.

Proof: Clearly the union of the A_α is connected so A, being the closure of this union, is connected. Hence A is a subcontinuum of M. Suppose that D is an m.c.o. of M such that D meets A. Then D open implies D meets the union of the A_α, so there exists an A_α of A such that D meets A_α. But A_α is a δ-subcontinuum of M so D is contained in A_α. Consequently D is contained in the union of the A_α and in A, the closure of this union. Thus we have seen that every m.c.o. D of M that meets A is contained in A. Now suppose that D is an m.c.o. of M such that F(D) is contained in A. Then, by hypothesis, the boundary of D meets A in a finite number of A_α. Let β be the maximum of the cardinal subscripts of these finite number of A_α. Then each of the finite number of A_α which meet F(D) is a subset of A_β, and therefore F(D) is a subset of A_β. Then D is contained in A_β, and hence in A, since A_β is a δ-subcontinuum of M. Thus every m.c.o. D of M whose boundary is contained in A is a subset of A. Consequently A is a δ-subcontinuum of M.

The following example shows that Theorem 9 minus the phrase "further suppose that M is such that if the boundary of an m.c.o. of M is contained in A then the boundary is contained in a finite number of A_α" no longer holds true.

<u>Example 1</u>: Let S be the Euclidean plane with the usual topology and let M be the circle (given in polar coordinates) r=a together with its interior. Clearly M is a compact continuum of S. Let $A_1=\{(r,\theta): 0\le\theta\le\frac{2\pi i}{i+1} \text{ and } r=a\}$. The A_1 con-

stitute a well-ordered monotonic ascending sequence of distinct δ-subcontinua of M. If A is the closure of the union of the A_i, then A is $\{(a,\theta):0\leq\theta\leq 2\pi\}$. Hence A is a subcontinuum of M that is not a δ-subcontinuum since Int(M) is an m.c.o. of M with F(Int(M)) contained in A, but Int(M) not contained in A.

Definition 2: If A is a connected subset of a compact continuum M, the δ-subcontinuum $\delta(A)$ of M is said to be δ-irreducible about A if A is a subset of $\delta(A)$ and if $\delta(A)$ is a subset of every δ-subcontinuum of M that contains A. (Clearly if such a δ-subcontinuum exists it is unique.)

To assert the existence of δ-irreducible δ-subcontinua we shall need to make use of the Brouwer Reduction Theorem. We state this theorem without proof since it is well known.

Theorem A (Brouwer Reduction Theorem): If (1) S is completely separable, (2) M is a compact subset of S which has property P, and (3) if the intersection of a monotonic descending sequence $\{A_\alpha\}$ of compact point sets has property P whenever each A_α has property P; then some closed subset of M is irreducible with respect to being closed and having property P.

Theorem 10: Let A be a connected subset of a compact continuum M. Further suppose that S is completely separable. Then there exists a δ-subcontinuum $\delta(A)$ of M such that $\delta(A)$

is δ-irreducible about A.

Proof: Let property P be the property of being a δ-subcontinuum of M that contains A. Clearly M itself has property P so the first two hypotheses of Brouwer Reduction Theorem are satisfied. But hypothesis (3) of the same theorem is satisfied also, as can be seen from the statement of Theorem 8. Consequently, by the Brouwer Reduction Theorem, there exists a δ-subcontinuum $\delta(A)$ of M such that $\delta(A)$ is δ-irreducible about A.

Note: In what follows when we use $\delta(A)$ it shall be assumed that the space S is completely separable, so we will be justified in asserting the existence of a $\delta(A)$ for any connected subset A of M.

Definition 3: If A is a connected subset of a compact continuum M, let A_1 be the union of all m.c.o. D of M such that D meets A, and let A_2 be the union of all m.c.o. E of M such that F(E) is contained in $Cl(A+A_1)$.

Example 2: Let S be the Euclidean plane with the usual topology. Figure 1 illustrates the sets A_1 and A_2 for the given connected set A. (It is to be noticed that an m.c.o. of M may belong simultaneously to A_1 and A_2.)

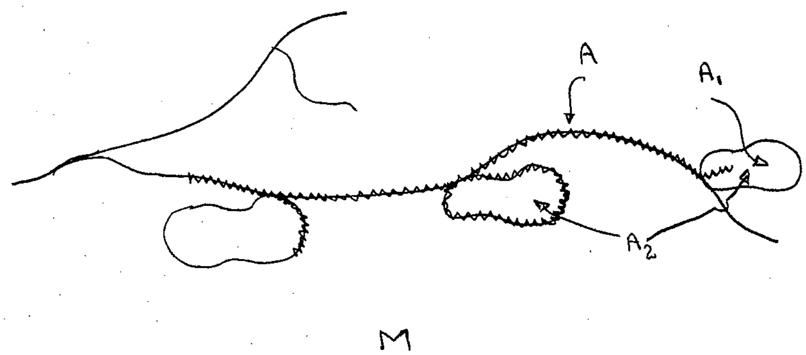

Figure 1

It appears that $\delta(A)$ and $Cl(A+A_1+A_2)$ coincide. The following lemma and theorem show that this is indeed the case.

<u>Lemma 4</u>: Let A be a connected subset of a compact continuum M. Then $Cl(A+A_1+A_2)$ is a δ-subcontinuum of M containing A.

<u>Proof</u>: We shall first show that $Cl(A+A_1+A_2)$ is a subcontinuum of M containing A. Clearly $Cl(A+A_1+A_2)$ is closed and contains A. Now $A+A_1$ is connected as otherwise $A+A_1=P+Q$ such that P and Q are mutually separated implies that A is contained in either P or Q since A is connected. Let us suppose the former so A is contained in P, and hence Q is contained in A_1. Then there exists a D belonging to A_1 such that D meets Q, so D must be a subset of Q since D is a connected subset of the union of the two mutually separated sets P and Q. Therefore D does not meet P, so D misses A. However D in A_1 implies D meets A. Consequently $A+A_1$, and also $Cl(A+A_1)$, is connected. If $Cl(A+A_1)+A_2$ is not connected then there exist two mutually separated sets P and Q such that $P+Q=Cl(A+A_1)+A_2$. Since $Cl(A+A_1)$ is connected, we may assume

that $Cl(A+A_1)$ is contained in P, so Q is contained in A_2. Then there exists an E belonging to A_2 such that E meets Q, and hence E must be a subset of Q since E is a connected subset of P+Q with P and Q mutually separated. Therefore F(E) is a subset of Cl(Q), so F(E) misses P and therefore also misses $Cl(A+A_1)$. However E in A_2 implies F(E) is contained in $Cl(A+A_1)$. Hence $Cl(A+A_1)+A_2$, and also $Cl(Cl(A+A_1)+A_2)=Cl(A+A_1+A_2)$, is connected. Consequently $Cl(A+A_1+A_2)$ is a subcontinuum of M that contains A.

We shall now show that $Cl(A+A_1+A_2)$ is a δ-subcontinuum of M.

Let D be an m.c.o. of M such that D meets $Cl(A+A_1+A_2)$. Then either (1) D meets A, or (2) D meets A_1, or (3) D meets A_2, since D is open. In (1), since D meets A, D must belong to A_1. In (2), there exists some D_1 in A_1 such that D meets D_1. Then $D+D_1$ is connected so the maximality of D_1 implies that D is contained in D_1, and hence is a subset of A_1. In (3), there exists some E in A_2 such that D meets E. Then, as in (1) above, D is contained in E so D is a subset of A_2. Hence (1), (2), and (3) all imply that D is a subset of $Cl(A+A_1+A_2)$, so every m.c.o. D of M that meets the set $Cl(A+A_1+A_2)$ is contained in that set.

Now suppose D is an m.c.o. of M such that $Cl(A+A_1+A_2)$ contains F(D). Let x be any element in F(D). If x belongs to $Cl(A_2)$, then for every open set V containing x, there exists an open connected set W containing x such that W is contained in V, W meets A_2, W meets D, and W meets S-D. Since

W meets A_2 there is an E belonging to A_2 such that WE is not empty. If DE is not empty then D+E is a connected open subset of M, so D+E=D=E by the maximality of both D and E. Then F(D)=F(E) so F(D) is contained in $Cl(A+A_1)$, and x belongs to $Cl(A+A_1)$. If DE is empty then D is contained in S-E, so (since W meets D) W meets S-E. Now W also meets E so W, being connected, meets F(E) by Lemma 1. But F(E) is contained in $Cl(A+A_1)$, so W meets $Cl(A+A_1)$ and then meets $A+A_1$. Hence V meets $A+A_1$ so x belongs to $Cl(A+A_1)$. Therefore F(D) is contained in $Cl(A+A_1)$ so D belongs to A_2, which in turn is a subset of $Cl(A+A_1+A_2)$. Consequently every m.c.o. D of M whose boundary is contained in the set $Cl(A+A_1+A_2)$ is itself contained in $Cl(A+A_1+A_2)$.

Thus, by Definition 1, $Cl(A+A_1+A_2)$ is a δ-subcontinuum of M.

__Theorem 11__: Let A be a connected subset of a compact continuum M. Then $\delta(A)$, the δ-subcontinuum of M that is δ-irreducible about A, is equal to $Cl(A+A_1+A_2)$.

__Proof__: Clearly $\delta(A)$ is contained in $Cl(A+A_1+A_2)$ since $\delta(A)$ is δ-irreducible about A and (Lemma 4) $Cl(A+A_1+A_2)$ is a δ-subcontinuum of M containing A. Conversely (1) A is a subset of $\delta(A)$, and (2) if D belongs to A_1 then D meets A, so D meets $\delta(A)$ and hence D is contained in $\delta(A)$ since $\delta(A)$ is a δ-subcontinuum of M. Thus A_1 is a subset of $\delta(A)$. Also if (3) E belongs to A_2, then F(E) belongs to $Cl(A+A_1)$. But

$Cl(A+A_1)$ by (1) and (2) is a subset of $\delta(A)$, so $F(E)$ is contained in $\delta(A)$. Hence E is a subset of $\delta(A)$ since $\delta(A)$ is a δ-subcontinuum. Consequently A_2 is a subset of $\delta(A)$, so (1), (2), and (3) imply that $Cl(A+A_1+A_2)$ is contained in $\delta(A)$. Therefore $\delta(A)=Cl(A+A_1+A_2)$.

Corollary 1: If A and B are connected subsets of a compact continuum M such that A is contained in B, then $\delta(A)$ is contained in $\delta(B)$.

Proof: Clearly A_1 is a subset of B_1 so $Cl(A+A_1)$ is a subset of $Cl(B+B_1)$. Then A_2 must be a subset of B_2 so $Cl(A+A_1+A_2)$ is contained in $Cl(B+B_1+B_2)$. Thus, by the theorem, $\delta(A)$ is contained in $\delta(B)$.

Corollary 2: If A is a connected subset of M then $\delta(\delta(A))=\delta(A)$, i.e. the δ operation applied to a connected subset of M is idempotent.

Proof: Since A is contained in $\delta(A)$ with both A and $\delta(A)$ connected, $\delta(A)$ is contained in $\delta(\delta(A))$ by Corollary 1. Now $\delta(\delta(A))$ is δ-irreducible about $\delta(A)$ so $\delta(\delta(A))$ is contained in $\delta(A)$ since $\delta(A)$ is a δ-subcontinuum of M containing $\delta(A)$. Thus $\delta(A)$ is the same as $\delta(\delta(A))$.

Corollary 3: If A is a δ-subcontinuum of M then A equals $\delta(A)$.

Proof: Since $\delta(A)$ is δ-irreducible about A and A is a δ-subcontinuum containing A we have $\delta(A)$ contained in A. Also the theorem implies A is a subset of $\delta(A)$. Consequently

A equals $\delta(A)$.

<u>Corollary 4</u>: Let A and B be connected subsets of M such that AB is connected. Then $\delta(AB)$ is a subset of $\delta(A)\delta(B)$, which in turn is a subset of $\delta(A)+\delta(B)$, with the latter set a subset of $\delta(A+B)$.

<u>Proof</u>: Corollary 1 implies $\delta(AB)$ is a subset of both $\delta(A)$ and $\delta(B)$ so $\delta(AB)$ is a subset of $\delta(A)\delta(B)$. Also A and B are connected subsets of the connected set A+B, so Corollary 1 implies $\delta(A)+\delta(B)$ is a subset of $\delta(A+B)$.

In general the set inclusions of Corollary 4 are proper, as can be seen from simple examples in the plane.

<u>Corollary 5</u>: Let f be a homeomorphism of M onto itself. If D is an m.c.o. of M then $f(\delta(D))$ equals $\delta(f(D))$, i.e. the δ and f operations commute.

<u>Proof:</u> By Definition 3, $f(Cl(D_2)=f(Cl\{$m.c.o. E of $M:F(E)$ is contained in $F(D)\})$. Since f is a homeomorphism the last mentioned set coincides with $N=Cl\{$m.c.o. $f(E)$ of $M:f(F(E))$ is contained in $f(F(D))\}$, with the $f(E)$ m.c.o. by Lemma 2. Then Lemma 3 implies that $N=Cl\{f(E):F(f(E))$ is contained in $F(f(D))\}$. But this set is $Cl(f(D)_2)$ since f is onto. Consequently $f(Cl(D_2))$ equals $Cl(f(D_2)$.

Now $f(\delta(D))$ equals (by the theorem) $f(D+Cl(D_2))$, which in turn equals $f(D)+Cl(f(D)_2)$ from the previous paragraph. But Theorem 11 also implies that $\delta(f(D))$ equals $f(D)+Cl(f(D)_2)$. Hence $f(\delta(D))$ equals $\delta(f(D))$.

Returning to Corollary 1 of the last theorem it appears as if we could induce a partial order on the δ-irreducible δ-subcontinua of M by defining $\delta(A)$ to be less than $\delta(B)$ if A is contained in B with A and B connected. The following example shows that this is impossible since $\delta(A)$ may be less than $\delta(B)$, $\delta(B)$ may be less than $\delta(A)$, but AB may be empty.

Example 3 (modification of the Lakes of Wada [2]): Let M, A_1, and B_1 be three rectangles together with their interiors in the plane S(with its usual topology) such that A_1 and B_1 are subsets of M(see Figure 2). In the first step of the construction we enlarge A_1 and B_1(keeping inside M) to form A_2 and B_2 such that the distance from every point of $M-(A_2+B_2)$ is less than one unit from every point of A_2+B_2. In the second step of the construction we enlarge A_2 and B_2 (staying within M) to form A_3 and B_3 such that the distance from every point of $M-(A_3+B_3)$ is less than one-half unit from every point of A_3+B_3. Continuing, at step i we enlarge A_{i-1} and B_{i-1} (without leaving M) to form A_i and B_i such that the distance from every point of $M-(A_i+B_i)$ is less than $\frac{1}{i}$ units from every point of A_i+B_i. Let A and B be the completed enlargements of the A_i and B_i, respectively. Then (1) M-(A+B) is a nowhere dense closed subset of S, (2) M-(A+B) is the common boundary of A,B, and S-(A+B), (3) A and B are distinct maximal connected open subsets of M, and (4) $\delta(A)=M=\delta(B)$.

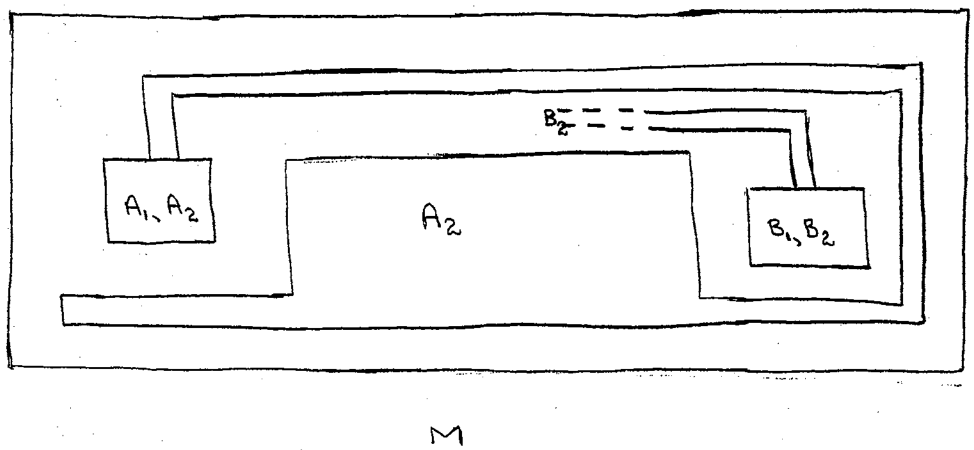

Figure 2

<u>Definition 4</u>: If A and B are intersecting δ-subcontinua of a compact continuum M then E(A,B) shall be the set of all m.c.o. E of M such that (1) F(E) is contained in A+B, (2) F(E) meets A-B, and (3) F(E) meets B-A.

We note that E(A,B) is disjoint from A+B since, if D belonged to E(A,B) and D met A+B, then D would be a subset of either A or B (as A and B are δ-subcontinua of M), so F(D) would be a subset of either A or B and condition (2) or (3) of Definition 4 would be violated.

<u>Lemma 5</u>: If A and B are intersecting δ-subcontinua of M then δ(A+B) equals A+B+Cl(E(A,B)).

<u>Proof</u>: Clearly A+B is connected so, by Theorem 11, $\delta(A+B)=Cl(A+B+(A+B)_1+(A+B)_2)$. Now $(A+B)_1=\{$m.c.o. D of M:D meets A+B$\}=\{$m.c.o. D of M:D meets A$\}+\{$m.c.o. D of M:D meets B$\}=A_1+B_1$. But A_1 is contained in A and B_1 is contained in B

by Corollary 3 to Theorem 11, so A_1+B_1 is a subset of A+B. Also $(A+B)_2=\{$m.c.o. E of M:F(E) is a subset of $A+B+(A+B)_1\}$ $=\{$m.c.o. E of M:F(E) is a subset of A+B$\}=\{$m.c.o. E of M:F(E) is a subset of A$\}+\{$m.c.o. E of M:F(E) is a subset of B$\}$+E(A,B) $=A_2+B_2+E(A,B)$, with A_2+B_2 contained in A+B by Corollary 3 to Theorem 11. Hence δ(A+B)=Cl(A+B+E(A,B))=A+B+Cl(E(A,B)) since both A and B are closed.

<u>Corollary 1:</u> A necessary and sufficient condition for the sum of two intersecting δ-subcontinua A and B to be a δ-subcontinuum is that Cl(E(A,B)) be empty.

<u>Corollary 2</u>: If we allow the empty set to be a δ-subcontinuum then a necessary and sufficient condition for the sum of two intersecting δ-subcontinua A and B to be a δ-subcontinuum is that (A+B)Cl(E(A,B)) be a δ-subcontinuum.

<u>Proof</u>: If A+B is a δ-subcontinuum then Cl(E(A,B)) is empty by Corollary 1, so (A+B)Cl(E(A,B)) is the empty set (a δ-subcontinuum by assumption). Conversely if (A+B)Cl(E(A,B)) is a δ-subcontinuum then Lemma 5 and Theorem 4 imply that A+B is a δ-subcontinuum.

<u>Lemma 6</u>: Let A,B, and C be δ-subcontinua of M such that none is a subset of any other and such that $AB\neq 0\neq AC$. Then $BC\neq 0$ if $\delta(A+B)=\delta(A+C)$.

<u>Proof:</u> By Lemma 5, δ(A+B)=A+B+Cl(E(A,B)) and δ(A+C)= A+C+Cl(E(A,C)) so A+B+Cl(E(A,B))=A+C+Cl(E(A,C)). If E(A,B) is empty then C is a subset of A+B so $BC\neq 0$. Therefore let us

suppose that $Cl(E(A,B))$ is not empty, and let E belong to $E(A,B)$. Now $\delta(A+B)=\delta(A+C)$ so E is a subset of $(C+Cl(E(A,C)))-A$ by the note following Definition 4. If E meets C then E is a subset of C, since C is a δ-subcontinuum, so $F(E)$ is a subset of C. Then $BC\neq 0$ since E is an element of $E(A,B)$. If E meets $Cl(E(A,C))$ then E meets $E(A,C)$ so E is an element of $E(A,C)$ by the maximality of E and the maximality of the elements of $E(A,C)$. Then $F(E)$ meets $C-A$ so there exists an element x belonging to $(C-A)F(E)$. But E belongs to $E(A,B)$ so $F(E)$ is contained in $A+B$. Consequently x must belong to $B-A$, so $BC\neq 0$.

The next two theorems are again concerned with the boundaries of δ-subcontinua of a compact continuum, and the proofs used depend heavily on Theorem 11.

Theorem 12: If D is an m.c.o. of a compact continuum M then $F(D)$ coincides with $F(\delta(D))$.

Proof: Clearly D is a subset of $\delta(D)$ so $F(D)$ is a subset of $F(\delta(D))$ by Corollary 1 of Theorem 6. It then suffices to show that $F(\delta(D))$ is a subset of $F(D)$. By Theorem 11, $\delta(D)=Cl(D+D_1+D_2)=Cl(D+D_2)$ since the maximality of D implies that D_1 is contained in D. Then $F(\delta(D))=F(Cl(D+D_2))$ with $F(Cl(D+D_2))$ being a subset of $F(D+D_2)$, and the last mentioned set being a subset of $F(D)+F(D_2)$. Hence $F(\delta(D))$ is a subset of $F(D)+F(D_2)$. Let us suppose that x is an element of $F(D_2)$. Then if V is any open connected set containing x (such a V exists since S is locally connected at x) V intersects D_2, so there exists E, an element of D_2, such that V intersects E. Now V also misses D_2 so V intersects S-E, and (by Lemma 1) V intersects F(E). But E is an element of D_2 so V intersects D (we showed that D_1 was a subset of D). Further, if V is contained in D then D intersects E, so D=E by the maximality of both D and E; but then E would not be an element of D_2. Hence V intersects S-D; so for any open connected set V containing x, $VD \neq 0 \neq V(S-D)$. Then x belongs to F(D), since given any open set W containing x there exists an open connected set V containing x with V contained in W. Consequently $F(\delta(D))$ is a subset of F(D); so F(D) coincides with $F(\delta(D))$.

<u>Corollary</u>: Let D be an m.c.o. of M. Then $\delta(D)=M$ if and only if $F(D)=F(M)$.

<u>Proof</u>: By the theorem $F(D)=F(\delta(D))$ with $\delta(D)$ a δ-subcontinuum of M. The conclusion then follows by Corollary 3 of Theorem 6.

In connection with the above corollary we note (by Theorem 11) that $\delta(F(M))=M$ if F(M) is connected.

The proof of Theorem 12 would have been trivial if $\delta(D)$ were equal Cl(D) for D an m.c.o. of M. That this is not true in general can be seen from Example 3 given earlier. There A was an m.c.o. of M such that $\delta(A)=M=Cl(A+B)$, but Cl(A) failed to meet B. Theorem 27 gives a sufficient condition in the plane for $\delta(D)$ to equal $Cl(D)$.

Even if $Int(M)$ is equal to a single m.c.o. D, we may have $\delta(D)$ a proper δ-subcontinuum of M. Further, if A is a connected subset of M, then $F(\delta(A))$ may not coincide with $F(A)$ (this is illustrated in Figure 3 below). Hence neither Theorem 12 nor its Corollary may be improved.

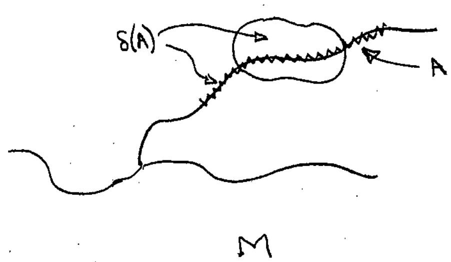

Figure 3

<u>Theorem 13</u>: Let A and B be intersecting δ-subcontinua of a compact continuum M. Then $F(\delta(A+B))$ coincides with $F(A+B)$.

<u>Proof</u>: Clearly $F(A+B)$ is contained in $F(A)+F(B)$, with A and B δ-subcontinua such that A and B are subsets of $\delta(A+B)$. Then, by Theorem 7, $F(A)+F(B)$ is a subset of $F(\delta(A+B))$. Hence $F(\delta(A+B))$ contains $F(A+B)$. Conversely let x be any element of $F(\delta(A+B))$, and V be any open set containing x. Then V meets $S-\delta(A+B)$ so (since A+B is a subset of $\delta(A+B)$) V meets $S-(A+B)$. Now, by Lemma 5, $F(\delta(A+B))=F(A+B+Cl(E(A,B)))$ so $F(\delta(A+B))$ is a subset of $F(A)+F(B)+F(Cl(E(A,B)))$, and x must then belong to this union. If x belongs to $F(A)+F(B)$ then V meets A+B.

If x belongs to $F(Cl(E(A,B)))$, let W be an open connected set containing x such that W is a subset of V. Then there exists an E belonging to E(A,B) such that W meets E. Also x in $F(\delta(A+B))$ implies W meets $S-\delta(A+B)$ so W meets S-E. By Lemma 1, W meets F(E) so (by condition (1) of Definition 4) W, and therefore V, meets A+B. Then $V(A+B)\neq 0 \neq V(S-(A+B))$ so x belongs to F(A+B). Consequently $F(\delta(A+B))$ is contained in F(A+B).

Corollary: Let A,B, and C be δ-subcontinua of M such that A meets B. Then C contained in $\delta(A+B)$ implies F(C) is contained in F(A+B).

Proof: By Theorem 7, F(C) is contained in $F(\delta(A+B))$. Then F(C) is contained in F(A+B) by the present theorem.

The next theorem is a generalization of a well-known theorem concerning the formation of subcontinua from mutually separated sets.

Theorem 14: Let A be a δ-subcontinuum of a compact continuum M, and let M-A be the sum of two mutually separated sets P and Q. Then A+P and A+Q are proper δ-subcontinua of M.

Proof: It is well known [3, p.108] that A+P and A+Q are both proper subcontinua of M. We shall now show that A+P is a δ-subcontinuum of M. (The proof that A+Q is a δ-subcontinuum is exactly the same.) Let D be an m.c.o. of M such that D meets A+P. If D meets A then D is a subset of A, since A is a δ-subcontinuum of M, so D is a subset of A+P. Otherwise DA=0 and (since M=A+P+Q) D is a subset of P+Q. Then D is a subset of either P or Q, since D is connected and P and Q are mutually separated sets. But D a subset of Q implies D misses A+P, in contradiction to hypothesis. Hence D is a subset of P and then D is a subset of A+P. Therefore D is a subset of A+P whenever D is an m.c.o. that meets A+P. Let us now suppose that D is an m.c.o. of M such that F(D) is a subset of A+P. Clearly M=A+P+Q so D must be a subset of A+P+Q. As shown in the first part of this proof, D is a subset of A+P if D meets A, and if D misses A then D is a subset of either P or Q. Now D a subset of Q implies F(D) is a subset of Cl(Q) so F(D) misses P, as P and Q are mutually separated sets. But F(D) is contained in A+P so F(D) must be wholly contained in A, and then D is contained in A since A is a δ-subcontinuum of M. Therefore D is a subset of A+P whenever D is an m.c.o. of M such that F(D) is a subset of A+P. Consequently A+P is a δ-subcontinuum of M.

CHAPTER II

δ-INDECOMPOSABLE AND δ-IRREDUCIBLE CONTINUA

We begin this chapter by showing that no indecomposable continuum in S may contain open sets.

Theorem 15: Let M be a continuum in our space S. Then (1) M decomposable implies there exists a proper subcontinuum A of M such that $Int_M(A)$, i.e. the interior of A with respect to M, is not empty; and (2) if there exists a proper subcontinuum A of M such that Int(A) is not empty then M is decomposable.

Proof: If M is decomposable then there exist proper subcontinua A and B of M such that $M=A+B$. Since B is proper, A-B is not empty so there is an element x belonging to A-B. Now B closed implies there exists an open set V (of S) containing x such that VB is null and VA is not null. But VM is an open set about x with respect to M, so x belongs to $Int_M(A)$ since $VM=VA$. Consequently $Int_M(A)$ is not empty; so (1) holds.

Let us now suppose that A is a proper subcontinuum of M such that Int(A) is not empty. Then Cl(M-A) is a proper subset of M. If M-A is connected so is Cl(M-A), and we have $M=A+Cl(M-A)$; so M is decomposable. If M-A is not connected then $M-A=P+Q$ such that P and Q are mutually separated sets. By Theorem 14, $M+P$ and $M+Q$ are proper subcontinua of M with $M=(M+P)+(M+Q)$; so M is again decomposable. Consequently conclusion (2) holds.

We wish to alter the concepts of continuum and of indecomposable continuum so that conclusion (2) of Theorem 15 may be false. We have already defined what we mean by δ-continuum, and we now define δ-indecomposable continuum. We shall then give an example of a δ-indecomposable continuum that possesses proper δ-subcontinua with interior points.

<u>Definition 5</u>: Let M be a compact continuum of S. If M cannot be expressed as the union of two of its proper δ-subcontinua then M is said to be δ-indecomposable. Otherwise M is said to be δ-decomposable.

We note that (1) if M is indecomposable then M is δ-indecomposable, and (2) (the contrapositive of (1)) if M is δ-decomposable then M is decomposable.

<u>Example 4</u>: In the Lakes of Wada example (Example 3) M is δ-indecomposable (for the proof see Theorem 16 below) but M possesses no proper δ-subcontinuum with interior points (relative to S, the plane).

<u>Example 5</u>: If we let M be the "chi" figure given in Figure 4 below and S be the plane, then M is δ-indecomposable (again see Theorem 16 below) and A_i, $i=1,2,3,4$, are proper

δ-subcontinua of M such that Int(A_i) is not empty for any i.

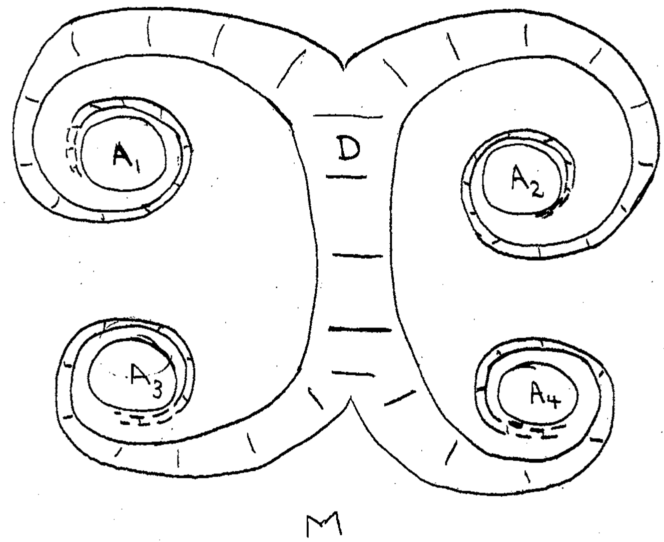

Figure 4

Clearly the existence of δ-indecomposable continua is assured since S is locally connected: for let x be any element of S and let V be an open connected set containing x; then M=Cl(V) is a δ-indecomposable continuum containing x (by virtue of Theorem 16) whenever M is compact.

We note that every non-degenerate compact continuum in E_1 (the Euclidean line with interval topology) is simultaneously decomposable and δ-indecomposable. This follows since every non-degenerate compact continuum in E_1 is a closed and bounded interval of the form $[a,b]$.

Theorem 16 [5, p.298]: Let M be a compact continuum of S. If there exists an m.c.o. D of M such that F(D) coincides with F(M), then M is δ-indecomposable.

Proof: Let us suppose that M is δ-decomposable so that there exist proper δ-subcontinua A and B of M such that M=A+B. Since D is a subset of M, either DA or DB is not empty. Let us suppose the former. By Corollary 1 to Theorem 6, F(D) is a subset of F(A). But F(D) coincides with F(M) so F(M) is a subset of F(A). Now A is a δ-subcontinuum of M so, by Theorem 5, F(A) is contained in F(M). Hence F(A) coincides with F(M) and A=M by Corollary 3 to Theorem 6. This is a contradiction to the hypothesis that A is proper, and therefore M must be δ-indecomposable.

The converse of Theorem 16 is not true in general. This is shown in the following example.

Example 6: Consider the indecomposable continuum M [4, p.424] and its image M' formed by mapping the arc $\frown{ab}$ of M onto a simple closed curve. We have Int(M)=D, an m.c.o. of M, but F(D) is a proper subset of F(M). However we shall see later (Corollary to Theorem 21) that M is δ-indecomposable.

Figure 5

<u>Theorem 17</u>: Let M be a δ-indecomposable continuum. If A is any proper δ-subcontinuum of M then M-A is a non-degenerate connected subset of M.

<u>Proof</u>: Clearly M-A is non-degenerate since (1) A is closed and (2) M is connected and closed. Let us suppose that M-A is not connected so M-A is the union of two mutually separated sets P and Q. By Theorem 14, A+P and A+Q are proper δ-subcontinua of M. But $M=(A+P)+(A+Q)$; so M is δ-decomposable. However we assumed that M was δ-indecomposable. Consequently M-A is connected.

<u>Corollary 1</u>: If A is any proper δ-subcontinuum of a δ-indecomposable continuum M then $\delta(M-A)$ coincides with M.

<u>Proof</u>: By Theorems 17 and 10 we know that $\delta(M-A)$ exists. Further if $\delta(M-A)$ is a proper δ-subcontinuum of M, then M is the union of the two proper δ-subcontinua A and $\delta(M-A)$; so M is δ-decomposable. We have a contradiction; so $\delta(M-A)$ must coincide with M.

<u>Corollary 2</u>: If A is any proper δ-subcontinuum of δ-indecomposable M then $F(\delta(M-A))$ equals $F(M)$.

<u>Corollary 3</u>: Let M be a δ-indecomposable continuum such that every m.c.o. of M has a non-degenerate boundary. Then no point of F(M) is a cut point of M.

<u>Proof</u>: Let x belong to F(M). Then $\delta(\{x\})=\{x\}$ since x cannot be the boundary of any m.c.o. of M. Consequently

$M-\{x\}$ is connected by the theorem, and hence x is not a cut point of M.

Corollary 3 of Theorem 17 does not hold for points of Int(M) since every interior point of an arc cuts the arc.

As a converse of Corollary 1 of Theorem 17 we have the following theorem.

Theorem 18: Suppose M is a compact continuum of S such that $\delta(M-A)=M$ for every proper δ-subcontinuum A of M (it being assumed that M-A is connected). Then M is δ-indecomposable.

Proof: If we assume that M is δ-decomposable then $M=A+B$, where A and B are proper δ-subcontinua of M. Hence M-A is a subset of B, so $\delta(M-A)$ is a subset of $\delta(B)$ by Corollary 1 to Theorem 11. Also by Corollary 3 of the same theorem, $\delta(B)$ coincides with B; so $\delta(M-A)$ is a subset of B. But $\delta(M-A)$ coincides with M, so B must equal M. This is contradictory to the hypothesis that B is proper. Consequently M is δ-indecomposable.

In order to simplify the proof of a converse to Theorem 16, let us consider the following definition and its characterization of δ-indecomposable compact continua as given in Theorem 19 below.

Definition 6: Let M be a compact continuum in S, and let A be a proper δ-subcontinuum of M. We shall say that A is a δ-subcontinuum of condensation of M if (1) M-A is connected, and if (2) every point of F(A) is a limit point of δ(M-A).

The δ-subcontinua A_i, i=1,2,3,4, in Figure 4 are δ-subcontinua of condensation of M. The proof of this statement is given in the following theorem.

<u>Theorem 19</u> [5, p.297]: A necessary and sufficient condition for a compact continuum M to be δ-indecomposable is that every proper δ-subcontinuum of M be a δ-subcontinuum of condensation of M.

<u>Proof</u>: We first show that M δ-indecomposable implies every proper δ-subcontinuum of M is a δ-subcontinuum of condensation of M. Let us suppose, by way of contradiction, that there is a proper δ-subcontinuum A of M that is not a δ-subcontinuum of condensation of M. Then there exists an element x belonging to F(A)-Cl(δ(M-A)), with M-A connected by Theorem 17. Consequently δ(M-A) is a proper subset of M. But this is a contradiction of Corollary 1 of Theorem 17. Hence every proper δ-subcontinuum of M is a δ-subcontinuum of condensation of M.

We now show that M must be δ-indecomposable if every proper δ-subcontinuum of M is a δ-subcontinuum of condensation of M. If we suppose that M is δ-decomposable, there exist two proper δ-subcontinua A and B of M such that M is the sum of A and B. By hypothesis, B is a δ-subcontinuum of condensation of M so every point of F(B) is a limit point of δ(M-B).

But M-B is a subset of A with M-B and A connected, so $\delta(M-B)$ is a subset of A by Corollaries 1 and 3 of Theorem 11. Then F(B) is contained in A so, by virtue of Theorem 7, B is contained in A. This implies that A is not proper; a contradiction to the hypothesis that M is δ-decomposable. Consequently M must be δ-indecomposable.

Theorem 20: Let D be an m.c.o. of a compact indecomposable continuum M. Then either (1) F(D) coincides with F(M), or (2) every point in F(D) is a limit point of $\delta(M-\delta(D))$.

Proof: Let us first suppose that $M-\delta(D)$ is not empty. Then $\delta(D)$ is a proper δ-subcontinuum of M. Therefore, by Theorem 19, every point of $F(\delta(D))$ is a limit point of $\delta(M-\delta(D))$. But Theorem 12 implies that $F(\delta(D))$ is the same as F(D), so every point of F(D) is a limit point of $\delta(M-\delta(D))$. Hence conclusion (2) holds.

If $M-\delta(D)$ is empty then $M=\delta(D)$; so F(M) coincides with F(D) by the Corollary to Theorem 12. Consequently conclusion (1) must hold.

We next show that, under very mild conditions, if A is an indecomposable subcontinuum then $\delta(A)$ is a δ-indecomposable subcontinuum. (The reader is asked to compare the following theorem with Theorem 25.)

Theorem 21: Let A be a connected subset of a compact continuum M such that whenever $\delta(A)$ is the sum of two δ-subcontinua then the intersection of at least one of these

δ-subcontinua with A must be connected. Then A indecomposable implies $\delta(A)$ must be δ-indecomposable.

Proof: Suppose $\delta(A)$ is δ-decomposable. Then $\delta(A)=B+C$ such that B and C are proper δ-subcontinua of $\delta(A)$. If $A=AB$ then A would be a subset of B and hence $\delta(A)$ would be contained in B. This is a contradiction so AB(and AC) is a proper subset of A, with $A=AB+AC$. Since A is indecomposable either AB or AC is not connected. Supposing the former, we have $AB=P+Q$ such that P and Q are mutually separated. By hypothesis A is indecomposable and AC is connected so A-AC (being contained in AB) is a subset of either P or Q. If A-AC is a subset of P then Cl(A-AC) is a proper subcontinuum of A, since Cl(A-AC) is contained in Cl(P) and QCl(P) is empty. Then $A=AC+Cl(A-AC)$ so A is decomposable. Hence A indecomposable implies $\delta(A)$ is δ-indecomposable.

Corollary [5, p.298]: Let M be a compact continuum such that whenever M is the sum of two δ-subcontinua then the boundary of at least one of these δ-subcontinua must be connected. Then M is δ-indecomposable whenever F(M) is indecomposable.

Definition 7: Let M be a compact continuum and let A be a proper δ-subcontinuum of M. We shall call A a maximal δ-subcontinuum of M if M contains no other proper δ-subcontinuum B such that A is a proper subset of B.

If A is a maximal δ-subcontinuum of M and B is any other proper δ-subcontinuum of M that meets A then either (1) B is a subset of A, or (2) $\delta(A+B)=M$.

Example 7: In Figure 6 below we have an example of a δ-decomposable compact continuum M in the plane such that (1) D is an m.c.o. of M, (2) F(D) is not connected, and (3) Cl(M-Cl(D)) is a maximal δ-subcontinuum of M.

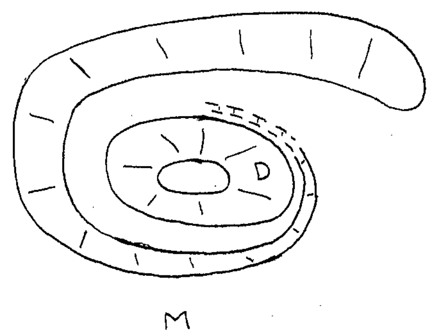

Figure 6

Example 8: By way of contrast Figure 7 is an example of a δ-decomposable continuum M in the plane such that M contains no maximal δ-subcontinuum.

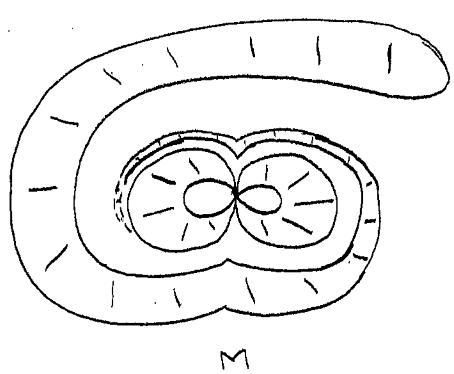

Figure 7

<u>Theorem 22</u>: Let M be a compact continuum with at least two disjoint maximal δ-subcontinua. Then M must be δ-indecomposable. (See [3, p.148] for the proof in the case of maximal subcontinua.)

<u>Proof</u>: Let P and R be two disjoint maximal δ-subcontinua of M. Suppose, by way of contradiction, that M is δ-decomposable. Then there exist two proper δ-subcontinua A and B of M such that A+B equals M. Since P is a subset of M, P meets A+B. Let us suppose that P meets A. By the comment following Definition 7, either (1) A is a subset of P, or (2) $\delta(A+P)=M$. But if A is a subset of P then (since A+B=M and PR=0) R is a subset of B, so A=P and B=R. Then M=P+R with P and R disjoint; contrary to the connectivity of M. Consequently $\delta(A+P)=M$. If R is a subset of B we have a contradiction in the same manner as before. Thus R must meet A. Again by Definition 7, either (1) A is a subset of R, or (2) $\delta(A+R)=M$. As before, A a subset of R implies M is not connected. Hence $\delta(A+R)=M$. Then by Lemma 6, $PR\neq 0$ since $\delta(A+P)=M=\delta(A+R)$. But P and R are disjoint. Therefore M is δ-indecomposable.

It is not know in general whether $\delta(M-A)$ is δ-indecomposable whenever A is a maximal δ-subcontinuum of a compact continuum M.

<u>Definition 8</u>: If M is a continuum, a δ-composant of M (with respect to some point p of M) is the set of all points

x of M such that there exists a proper δ-subcontinuum of M containing both x and p.

Any δ-composant is the union of connected sets all having a point p in common. Consequently any δ-composant is connected.

Depending upon the location of p in M there may or may not be a δ-composant of M with respect to p. In Example 7 there is a δ-composant for every point of M; but in Example 5 no δ-composant exists for points in D.

Theorem 23: If the continuum M is δ-decomposable then M is a δ-composant for some point of M.

Proof: If M is δ-decomposable then M is the sum of two proper δ-subcontinua A and B of M. Since M is connected and A and B are closed there exists a point p belonging to both A and B. It is then clear that M is the δ-composant of M with respect to p.

Theorem 24: If the compact continuum M is δ-decomposable then every two δ-composants intersect.

Proof: Let A_p and A_q be two different δ-composants of M with respect to p and q respectively. Since M is δ-decomposable there exist δ-subcontinua B and C of M such that $B+C=M$. We may suppose that p belongs to B so B is contained in A_p, since B is a proper δ-subcontinuum of M containing p. If q belongs to B then B is contained in A_q (proof as before) and hence A_pA_q is not empty. Otherwise q belongs to C and C is contained in A_q, so M connected implies BC is not empty and hence A_pA_q is not empty.

The following two theorems on δ-composants may be found in the paper by O.H. Hamilton [5, p.297-298] and are valid when S is Euclidean n-space, and M, a compact continuum, is not the sum of a finite number (greater than one) of its δ-subcontinua $g_1, \ldots, g_k$ such that, for each integer i, every boundary point of g_i is a limit point of $M-g_i$.

<u>Theorem B</u>: Every δ-composant of M is the sum of a countable number of δ-subcontinua of M.

<u>Theorem C</u>: If M is δ-indecomposable and no m.c.o. D of M is such that F(D) equals F(M), then no two δ-composants of M intersect.

To conclude this chapter we return to the Brouwer Reduction Theorem and Definition 2, and say that a δ-subcontinuum A of a compact continuum M is δ-irreducible with respect to property P if A has property P and no proper δ-subcontinuum of A has property P.

<u>Theorem 25</u>: Let A be a subcontinuum of a compact continuum M such that A is irreducible with respect to some property P. Then $\delta(A)$ is δ-irreducible with respect to P if $\delta(A)$ enjoys property P.

<u>Proof</u>: Let N be δ-irreducible with respect to property P. (We are assuming that P is such that the hypotheses of the Brouwer Reduction Theorem are satisfied, so N does exist.)

Since N is a subcontinuum of M enjoying property P, we have A contained in N by the irreducibility of A. Then $\delta(A)$ is contained in N by Corollaries 1 and 3 of Theorem 11. But $\delta(A)$ possessing property P implies N is contained in $\delta(A)$ by the δ-irreducibility of N. Consequently N coincides with $\delta(A)$, so $\delta(A)$ is δ-irreducible with respect to P.

In general the converse of Theorem 24 is not true, as the following example shows.

Example 9: Let P be the property "containing both the points a and b". Then, in Figure 8 below, B+C is not irreducible with respect to P, but $\delta(B+C)=M$ is δ-irreducible with respect to P. (In the figure, B is the arc from a to b and C is the arc from b to c.)

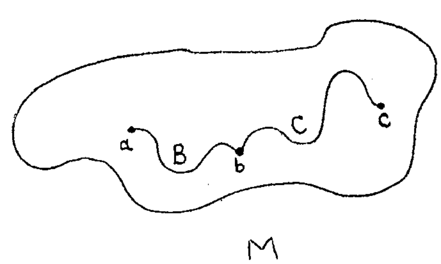

Figure 8

CHAPTER III

APPLICATIONS TO EUCLIDEAN n-SPACE

In this chapter we shall restrict S, the embedding space, to be Euclidean n-space, E_n.

We first show that any compact convex subset of S with interior is δ-indecomposable.

<u>Theorem 26</u>: If M is a convex body lying in E_n then M is δ-indecomposable.

<u>Proof</u>: We know [6, p.30] that Int(M) is convex. Then Int(M) is connected [6, p.74] so Int(M) is an (in fact, the only) m.c.o. of M. It is an easy matter to show that F(Int(M)) equals F(M). Hence M=Int(M)+F(Int(M)) is δ-indecomposable by Theorem 16.

<u>Corollary</u>: If M is a compact convex subset of E_n such that M contains the origin and n linearly independent vectors, then M is δ-indecomposable.

<u>Proof</u>: It is well known that any convex subset of E_n containing the origin and n linearly independent vectors has interior points relative to E_n. The conclusion then follows from the theorem.

Most of the theorems in this chapter are restricted to

E_2, the plane, and require that certain subsets do not separate E_2. The usefulness of these restrictions is quite apparent in Theorem 28 below.

Theorem 27: Let M be a compact continuum in the plane. A sufficient condition for $\delta(D)$ to coincide with Cl(D), for every m.c.o. D of M, is that no m.c.o. subset of M together with its boundary, separate the plane.

Proof: Let D and E be distinct m.c.o. of M such that D contains F(E). Clearly S-Cl(D)=E+(S-Cl(D+E)). We wish to show that the last two mentioned sets are mutually separated. Now (Cl(E))(S-Cl(D+E))=(Cl(E))(S-(Cl(D)+Cl(E)) with the latter set contained in the empty set (Cl(E))(S-Cl(E)). Hence Cl(E) misses S-Cl(D+E). If x belongs to the intersection of E and Cl(S-Cl(D+E)) then E meets S-Cl(D+E) (since E is an open set about x) so E meets S-E (Since S-Cl(D+E) is contained in S-E). This is a contradiction so E fails to meet Cl(S-Cl(D+E)). Hence E and S-Cl(D+E) are mutually separated sets and Cl(D) separates S. This violates the hypothesis of the theorem so D and E must coincide. Then δ(D) equals Cl(D) by Theorem 11.

Under the hypothesis of Theorem 27, Cl(D) not separating the plane implies δ(D) doesn't separate the plane, since Cl(D)=δ(D). The converse of this statement, however, is not true in general as can easily be seen from Figure 4.

The following theorem strengthens the conclusion of

Theorem 20.

Theorem 28: Let M be a compact continuum in the plane such that no m.c.o. subset of M, together with its boundary, separates the plane. If M is δ-indecomposable and there is no m.c.o. D of M such that F(D) equals F(M), then for any m. c.o. E of M with connected boundary every point of F(E) is a limit point of δ(M-E).

Proof: By virtue of Theorem 20 it suffices to show that M-E is connected. Now M-E=(M-Cl(E))+(Cl(E)-E), with Cl(E)-E connected by hypothesis. Also M-Cl(E) is connected by Theorem 17 (Cl(E) equals δ(E) by Theorem 27) so Cl(M-Cl(E)) is connected. Clearly Cl(M-Cl(E)) is a subset of M-E, so M-E=Cl(M-Cl(E))+(Cl(E)-E) with the latter two sets both being connected and having a non-empty intersection. Hence M-E is connected.

The following theorem reduces the hypotheses needed in Corollary 3 of Theorem 17.

Theorem 29: Let M be a compact continuum in the plane such that every m.c.o. of M is simply connected. Then M δ-indecomposable implies no point of F(M) cuts M.

Proof: We know [7, p.256] that any simply connected open subset of C(the complex numbers), distinct from C, has a non-degenerate boundary. The conclusion then follows from Corollary 3 of Theorem 17.

We have the following characterization theorem for an m.c.o. of a compact continuum in the plane.

<u>Theorem 30</u>: If D and E are distinct m.c.o. of a compact continuum in the plane such that $\delta(E)$ contains D, then F(D) is connected.

<u>Proof</u>: By Theorems 6 and 12, F(D) is contained in F(E), with D and E disjoint open connected sets in the plane. Further F(D) is bounded since M is compact. Hence [8, p.189] F(D) is connected.

<u>Corollary 1</u>: If D and E are distinct m.c.o. of a compact continuum in the plane such that $\delta(E)$ contains D, then D is simply connected.

<u>Proof</u>: In the plane, D is simply connected whenever F(D) is connected, so D is simply connected by the theorem.

<u>Corollary 2</u>: If M is a compact continuum in the plane with two distinct m.c.o. D and E such that $\delta(D)$ coincides with $\delta(E)$, then (1) F(D)=F(E), a connected set, (2) D and E are both simply connected, and (3) both Cl(D) and Cl(E) separate the plane. (For an example of such a situation see Figure 2.)

<u>Proof</u>: Clear by Theorem 30, its Corollary, Theorem 12, and Theorem 27.

<u>Corollary 3</u>: Let M be a compact continuum in the plane and let D be an m.c.o. of M with non-connected boundary. Then

F(E) is a proper subset of F(M) for every m.c.o. E of M distinct from D. Consequently if there are two (or more) m.c.o. of M with non-connected boundaries, then no m.c.o. G of M is such that F(G)=F(M).

Proof: If there is an m.c.o. E of M distinct from D such that F(E) equals F(M) then, by the Corollary to Theorem 12, δ(E) equals M. Hence D is a subset of δ(E) so, by the theorem, F(D) is connected. This is contrary to hypothesis; so F(E) is a proper subset of F(M) for every m.c.o. E of M different from D.

Corollary 3 to Theorem 30 has interesting applications in connection with δ-indecomposable continua in the plane. We explore these applications in the following theorem.

Theorem 31: If M is a compact continuum in Euclidean n-space with connected boundary, then every m.c.o. of M has a connected boundary.

Proof: Since F(M) is connected, F(M) is a continuum. Hence [9, p.343] every complementary domain of F(M) has connected boundary. It is easy to see that every m.c.o. of M is a complementary domain of F(M); so every m.c.o. has connected boundary.

In Corollaries 1 and 2 below it is assumed that M is a compact continuum in the plane that is not the sum of a finite number (greater than one) of its δ-subcontinua $g_1, g_2, \ldots, g_k$ such that, for each integer i, every boundary point of g_i is a limit point of $M-g_1$.

<u>Corollary 1</u>: If there is an m.c.o. D of M such that F(D) is not connected, then $\delta(D)$ must equal M whenever M is δ-indecomposable.

<u>Proof:</u> We know [5, p.298] that either F(M) is indecomposable or there is an m.c.o. E of M such that $\delta(E)$ equals M. Now F(D) is not connected so F(M) can not be connected (by the theorem), and obviously can not be indecomposable. Hence there is an m.c.o. E of M with $\delta(E)=M$. If E is distinct from D then, by Corollary 3 to Theorem 30, $\delta(E)$ cannot equal M (since, as we have seen before $\delta(E)=M$ implies $F(E)=F(\delta(E))=F(M)$). Therefore $\delta(D)$ must equal M.

<u>Corollary 2</u>: If M has at least two m.c.o. with non-connected boundaries then M is δ-decomposable.

<u>Proof</u>: Clear by Corollary 1 above and Corollary 3 to Theorem 30.

We end this exposition with some remarks on unsolved problems and conjectures. It is hoped that these remarks will serve as a starting place for further work on δ-continua.

If A and B are intersecting δ-subcontinua of a compact continuum M, under what conditions is $\delta(A+B)$ the same as A+B? In the light of Lemma 5 this amounts to finding conditions under which E(A,B) is null. If M is a subset of the plane and D is an m.c.o. of M such that (1) D is a Jordan domain (i.e. $F(D)$ is a simple closed curve), and (2) $ABF(D)$ is connected, then D cannot possibly belong to $E(A,B)$. This is, of course, a very special situation. Examples, similar to Example 8 given previously, may be constructed where D belongs to $E(A,B)$ if (1) $Cl(D)$ separates the plane, or (2) $ABF(D)$ is not connected. Furthermore, all examples obtained possessed either property (1) or property (2). Therefore we have the following conjecture: If A and B are intersecting δ-subcontinua of a compact continuum M and D is an m.c.o. of M such that $Cl(D)$ does not separate the space S and such that $ABF(D)$ is connected then D does not belong to $E(A,B)$. Putting this in a slightly different form we have: If A and B are intersecting δ-subcontinua of a compact continuum M such that for every m.c.o. D of M whose boundary is a subset of $A+B$, $Cl(D)$ does not separate S and $ABF(D)$ is connected, then $\delta(A+B)=A+B$.

Many theorems dealing with indecomposable continua would be true for δ-indecomposable continua (and exactly the same proofs could be used) having the property that $\delta(A+B)$ equals $A+B$. Hence the desirability of having $\delta(A+B)=A+B$ is apparent.

It is easy to see that M cannot be hereditarily δ-decomposable if $Int(M)$ is not empty. Does the same conclusion hold if M is hereditarily δ-indecomposable? In other words, if M is hereditarily δ-indecomposable is M hereditarily indecomposable? Also if $F(M)$ is hereditarily indecomposable is $Int(M)$ empty?

The answers to these questions, even restricted to the case where S is the plane, would be interesting additions to the theory of δ-continua.

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APPENDIX

NUMBERED DEFINITIONS, RESULTS, AND EXAMPLES

VITA

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